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Partial eigenvalue assignment for structural damage mitigation

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ABSTRACT

In partial eigenvalue assignment, not all eigenvalues of the open loop system matrix are modified through a multiple input state or output feedback controller. This freedom available to assign selected eigenvalues of the closed loop system matrix has been widely used in design contexts such as to eliminate spillover effects in structural control problems. Similar approach is also required to modify damping and/or stiffness characteristics in selected eigenmodes of a damaged structure. When an external force acts on the damaged structure, partial eigenvalue assignment in this fashion will attempt to use minimal control effort and keep the structure active with safe operation. In this paper, a new approach to partial eigenvalue assignment and its application to structural damage mitigation are presented. A three mass spring–damper model with damage in one of the springs is illustrated with damping modifications at specific eigenmodes. The procedure is repeated for a second example, which is a cantilever beam modeled using two inputs and 10 state variables.

1. Introduction

Significant progress has been made in the areas of structural health monitoring where the objective is not only to detect damages but also to determine the status of the structure [1,2]. The status of the structure is generally inferred through the new stiffness matrix associated with the damage [3–7]. Depending upon an assessment made in damage prognosis studies [8], a structure with less severe damage is often required to be operational in a clean as well as in an uncertain environment. In these cases, the dynamic response of the damaged structure is expected to operate with sufficient damping and stiffness characteristics incorporated in the structure. Such an operational structure with damage mitigated using an integrated sensor, actuator and controller architecture often gives better life to the structure.

In terms of the closed loop pole constraints, a controller for damage mitigation will modify the eigenvalues of the damaged structure in the sense of a partial eigenvalue assignment (PEA) algorithm [9–12]. If the open loop eigenvalues of the damaged structure exhibit insufficient damping and/or stiffness, they are required to be modified through a feedback controller. These altered eigenvalues enhance the damping and stiffness requirements that would have deteriorated in the damaged structure. Further, since stiffness perturbation is local and since it normally confines to a damage location, the unaltered eigenvalues are expected to exhibit sufficient damping and stiffness requirements to the structure. Thus, the objective of the feedback controller in damaged structure is to restore the damping and stiffness characteristics through PEA. In this paper, a new design procedure to synthesize such a feedback controller using PEA suitable for damage mitigation is presented.

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PEA has been a subject of immense interest to design controllers that avoid spillover problems in structures [12]. The freedom available to assign eigenvector elements in multiple input systems has also been used for robustness enhancement due to parameter perturbations [10, 13–15]. In this paper, linear algebraic methods [16] are used for PEA in first order systems. A procedure to choose eigenvectors corresponding to the desired eigenvalues is presented. The eigenstructure is used to determine the state and output feedback controller such that the eigenvalues of the closed loop system matrix are the spectrum of the open loop matrix eigenvalues and the desired eigenvalues. The design procedure is illustrated for damage mitigation in structures.

In Section 2, problem formulation is discussed. In Section 3, PEA using linear algebraic methods is presented. Section 4 develops a procedure to design state and output feedback controllers. In Section 5, structural damage mitigation using a spring–mass model is illustrated. Conclusions are presented in Section 6.

2. Problem formulation

Consider a finite element model of the discrete or continuous structure of the form

$$M\ddot{\mathbf{q}}+C\dot{\mathbf{q}}+\mathrm{K}\mathrm{q}=\mathbf{D}u \tag{1}$$

where $\ddot{\mathbf{q}}(t)$ is an n-component vector representing the acceleration of the structural degrees of freedom assumed at the nodes, $\dot{\mathbf{q}}(t)$ is the velocity vector and $\mathbf{q}(t)$ is the position vector. $u(t)$ is an m-component control vector. $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the compatible mass, damping and stiffness matrices of order n by n. $\mathbf{D}$ is the control influence matrix of order n by m. Assume $1<m\leq 2n$. Suppose the health monitoring techniques determine the status of the structure to be damaged. Then the stiffness matrix begins to deteriorate as a function of uncertain parameters α_i giving rise to the new stiffness matrix

$$\hat{\mathbf{K}}=\mathbf{K}+\sum_{i=1}^{r}\alpha_i\mathbf{K}_i$$

Here $\mathbf{K}_i$ are constant structured matrices. In continuous structure, usually they are tri-diagonal. In discrete structures, they are sparse. Further, r is the number of finite elements assumed to be damaged in the modeling. In a discrete spring–mass–damper model, r refers the number of springs that are damaged. Assuming a single damage, the new stiffness matrix can be estimated as

$$\hat{\mathbf{K}}=\mathbf{K}+\alpha_1\mathbf{K}_1$$

Similarly, without loss of generality, the damping matrix can be estimated as

$$\hat{\mathbf{C}}=\mathbf{C}+\beta_1\mathbf{C}_1$$

Let $\mathbf{q}'$ denotes the transpose of the n-component column vector $\mathbf{q}$. Defining the $2n$-component vector $\mathbf{x}(t)$ as $\mathbf{x}=[\mathbf{q}',\dot{\mathbf{q}}']'$, the first order state-space model for the damaged structure becomes

$$\dot{\mathbf{x}}=(\mathbf{A}+\mathbf{E})\mathbf{x}+\mathbf{B}\mathbf{u} \tag{2}$$

where

$$\mathbf{A}=\begin{bmatrix}0_{(n)} & \mathbf{I}_{(n)}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\in R^{2n\times 2n}$$

$$\mathbf{E}=\alpha_1\begin{bmatrix}0_{(n)} & 0_{(n)}\\ -\mathbf{M}^{-1}\mathbf{K}_1 & 0_{(n)}\end{bmatrix}+\beta_1\begin{bmatrix}0_{(n)} & 0_{(n)}\\ 0_{(n)} & -\mathbf{M}^{-1}\mathbf{C}_1\end{bmatrix}\in R^{2n\times 2n}$$

and

$$\mathbf{B}=\begin{bmatrix}0_{(n\times m)}\\ \mathbf{M}^{-1}\mathbf{D}\end{bmatrix}\in R^{2n\times m}$$

Depending upon the damage assessment which is based on the values estimated for α_1 and β_1, structural damage mitigation constantly ensures adequate damping and stiffness augmentation to some of the eigenmodes of the structures where they are deteriorated due to damage. The rest of the eigemodes with unaltered eigenvalues are assumed to have the necessary damping and stiffness characteristics. In this framework, PEA technique proposed in this paper considers a completely controllable system in Eq. (2) and alters some of its eigenvalues through a feedback controller. That is, if λ_p, $p=1,2,\ldots,2n$ are the complex conjugate open loop eigenvalues of the matrix $\mathbf{A}$, then the altered complex conjugate eigenvalues of the matrix $(\mathbf{A}+\mathbf{E}-\mathbf{BG})$ are denoted by $\hat{\lambda}_i$, where $i=1,2,\ldots,\overline{p}$. The unaltered complex conjugate eigenvalues of the matrix $(\mathbf{A}+\mathbf{E}–\mathbf{BG})$ by the feedback controller $\mathbf{G}$ are denoted by λ_k, where $k=\overline{p}+1,\overline{p}+2,\ldots,2n$. In the next section, the PEA technique for state and output feedback cases is presented. Thus the damping and stiffness characteristics of the damaged system are modified at selective eigenmodes.

3. Linear algebraic methods in PEA

Given a set of p sensor measurements $\mathbf{y}=\mathbf{Cx}$ and a control law in output feedback format $u=-\mathbf{Gy}$, the closed loop system matrix is written as $\mathbf{A}_c=(\mathbf{A}+\mathbf{E}-\text{BGC})$ or $\mathbf{A}_c=(\hat{\mathbf{A}}-\text{BGC})$, where $\hat{\mathbf{A}}=\mathbf{A}+\mathbf{E}$. Let the eigenvector corresponding to $\hat{\lambda}_i=\hat{\lambda}_{i,R}+j\hat{\lambda}_{i,I}$ be $\hat{\mathbf{v}}_i \in C^{2n}$. Similarly for $\lambda_k=\lambda_{k,R}+j\lambda_{k,I}$, the eigenvectors are $\mathbf{v}_k \in C^{2n}$. Denote the real and imaginary parts of the eigenvectors as $\hat{\mathbf{v}}_i=\hat{\mathbf{v}}_{i,R}+j\hat{\mathbf{v}}_{i,I}$ and $\mathbf{v}_k=\mathbf{v}_{k,R}+j\mathbf{v}_{k,I}$, respectively. Let

$$\text{GC}\hat{\mathbf{v}}_{i,R}=\hat{\mathbf{w}}_{i,R} \;\text{ and }\; \text{GC}\hat{\mathbf{v}}_{i,I}=\hat{\mathbf{w}}_{i,I} \tag{3}$$

where $\hat{\mathbf{w}}_{i,R},\hat{\mathbf{w}}_{i,I} \in R^m$. Then the eigenvalue and eigenvector constraint for the altered eigenvalues becomes

$$\begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} \hat{\mathbf{w}}_{i,R} \\ \hat{\mathbf{w}}_{i,I} \end{pmatrix}=\begin{pmatrix} -(\hat{\lambda}_{i,R}\mathbf{I}_{(2n)}-\hat{\mathbf{A}}) & \hat{\lambda}_{i,I}\mathbf{I}_{(2n)} \\ -\hat{\lambda}_{i,I}\mathbf{I}_{(2n)} & -(\hat{\lambda}_{i,R}\mathbf{I}_{(2n)}-\hat{\mathbf{A}}) \end{pmatrix}\begin{pmatrix} \hat{\mathbf{v}}_{i,R} \\ \hat{\mathbf{v}}_{i,I} \end{pmatrix} \tag{4a}$$

which can be written as

$$\overline{\mathbf{B}}\hat{\mathbf{w}}_i=\hat{\mathbf{F}}\hat{\mathbf{v}}_i, \quad i=1,2,\ldots,\overline{p} \tag{4b}$$

Likewise, the eigenvalue–eigenvector constraint for the unaltered eigenvalues is

$$\begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{w}_{k,R} \\ \mathbf{w}_{k,I} \end{pmatrix}=\begin{pmatrix} -(\lambda_{i,R}\mathbf{I}_{(2n)}-\hat{\mathbf{A}}) & \lambda_{i,I}\mathbf{I}_{(2n)} \\ -\lambda_{i,I}\mathbf{I}_{(2n)} & -(\lambda_{i,R}\mathbf{I}_{(2n)}-\hat{\mathbf{A}}) \end{pmatrix}\begin{pmatrix} \mathbf{v}_{k,R} \\ \mathbf{v}_{k,I} \end{pmatrix} \tag{5a}$$

$$\overline{\mathbf{B}}\mathbf{w}_k=\mathbf{F}\mathbf{v}_k, \quad k=\overline{p}+1,\overline{p}+2,\ldots,2n \tag{5b}$$

The dimensions of the matrix $\hat{\mathbf{F}}$ and $\mathbf{F}$, respectively, are $4n$ by $4n$. In the altered eigenvalue case, matrix $\hat{\mathbf{F}}$ is non-singular. In the unaltered eigenvalue case, matrix $\mathbf{F}$ is singular. In both situations, we are interested in finding the non-zero vectors $\{\mathbf{v}_k,\mathbf{w}_k\}$ and $\{\hat{\mathbf{v}}_i,\hat{\mathbf{w}}_i\}$, respectively. These vectors are used to compute the gain matrix $\mathbf{G}$ from Eq. (3).

One of the results in linear algebra states that the system of equations $\overline{\mathbf{A}}\overline{\mathbf{x}}=\overline{\mathbf{b}}$ is solvable if $\overline{\mathbf{b}}'\mathbf{y}=0$ for all $\mathbf{y}$ satisfying $\overline{\mathbf{A}}'\mathbf{y}=0$ [16]. In this result, if $\overline{\mathbf{A}}$ is a square matrix, singularity of $\overline{\mathbf{A}}$ is implicitly assumed. Thus the result for non-singular $\overline{\mathbf{A}}$ (or) non-square $\overline{\mathbf{A}}$ requires further investigation. In the non-singular case, although $\overline{\mathbf{x}}$ is unique, it resides in the null space of $\mathbf{Y}'\overline{\mathbf{A}}$ where the column vectors of $\mathbf{Y}$ is orthogonal to $\overline{\mathbf{b}}$. In the present case, consider Eq. (4b)

$$\overline{\mathbf{B}}\hat{\mathbf{w}}_i=\hat{\mathbf{F}}\hat{\mathbf{v}}_i=\hat{\mathbf{b}}_i$$

Here, $\hat{\mathbf{w}}_i$ is solvable if $\hat{\mathbf{b}}_i$ is orthogonal to the vectors $\boldsymbol{\eta}_i \in R^{4n}$ that are again orthogonal to the column vectors of $\overline{\mathbf{B}}$. Thus,

$$\boldsymbol{\eta}'\hat{\mathbf{F}}\hat{\mathbf{v}}_i=0$$

where $\boldsymbol{\eta}=[\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\ldots,\boldsymbol{\eta}_{\ell_1}]$ and ℓ_1 is the number of vectors forming a basis for the null space of $\overline{\mathbf{B}}'$. The order of the matrix $\boldsymbol{\eta}'\hat{\mathbf{F}}$ is $\ell_1 \times n$. Further, $\hat{\mathbf{v}}_i$ resides in the null space of $\boldsymbol{\eta}'\hat{\mathbf{F}}$ with basis $\hat{\boldsymbol{\eta}}_1,\hat{\boldsymbol{\eta}}_2,\ldots,\hat{\boldsymbol{\eta}}_{\ell_2}$ where ℓ_2 is the dimension of the null space of $\boldsymbol{\eta}'\hat{\mathbf{F}}$. Let

$$\hat{\mathbf{v}}_i=\hat{c}_1\hat{\boldsymbol{\eta}}_1+\hat{c}_2\hat{\boldsymbol{\eta}}_2+\cdots+\hat{c}_{\ell_2}\hat{\boldsymbol{\eta}}_{\ell_2}=[\hat{\boldsymbol{\eta}}_1 \;\; \hat{\boldsymbol{\eta}}_2 \;\; \ldots \;\; \hat{\boldsymbol{\eta}}_{\ell_2}]\begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \vdots \\ \hat{c}_{\ell_2} \end{bmatrix}=\hat{\boldsymbol{\eta}}\hat{\mathbf{c}}$$

$\hat{\mathbf{c}}_i$ are the non-zero constants which can be determined from the following eigenvalue and eigenvector constraint

$$\overline{\mathbf{B}}\hat{\mathbf{w}}_i=\hat{\mathbf{F}}\hat{\mathbf{v}}_i \Rightarrow \left[\overline{\mathbf{B}}-\hat{\mathbf{F}}\hat{\boldsymbol{\eta}}\right]\left[\hat{\mathbf{w}}_i\hat{\mathbf{c}}\right]=0 \tag{6}$$

Thus, the non-zero constants $\hat{\mathbf{w}}_i$ and $\hat{\mathbf{c}}_i$ reside in the null space of $\left[\overline{\mathbf{B}} \;\; -\hat{\mathbf{F}}\hat{\boldsymbol{\eta}}\right]$ and they exactly determine $\hat{\mathbf{w}}_i$ and $\hat{\mathbf{v}}_i$.

Similar procedure is repeated for the unaltered eigenvalues, for which the eigenvalue–eigenvector constraint is given by

$$\overline{\mathbf{B}}\mathbf{w}_k=\mathbf{F}\mathbf{v}_k=\mathbf{b}_k.$$

However, now $\mathbf{F}$ is singular. $\mathbf{w}_k$ is solvable if $\boldsymbol{\eta}'\mathbf{F}\mathbf{v}_k=0$. This time $\mathbf{v}_k$ reside in the null space of $\boldsymbol{\eta}'\mathbf{F}$ with basis $\tilde{\boldsymbol{\eta}}_1,\tilde{\boldsymbol{\eta}}_2,\ldots,\tilde{\boldsymbol{\eta}}_{\ell_3}$, where ℓ_3 represents its dimension. Let

$$\mathbf{v}_k=c_1\tilde{\boldsymbol{\eta}}_1+c_2\tilde{\boldsymbol{\eta}}_2+\cdots+c_{\ell_3}\tilde{\boldsymbol{\eta}}_{\ell_3}=[\tilde{\boldsymbol{\eta}}_1 \;\; \tilde{\boldsymbol{\eta}}_2 \;\; \ldots \;\; \tilde{\boldsymbol{\eta}}_{\ell_3}]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{\ell_3} \end{bmatrix}=\tilde{\boldsymbol{\eta}}c$$

where $\mathbf{c}_k$ are the non-zero constants to be determined.

Again, since $\mathbf{F}$ is singular, $\mathbf{v}_k$ is solvable if and only if $\mathbf{b}_k$ is orthogonal to vectors $\boldsymbol{\delta}_i \in R^{4n}$ that are orthogonal to the column vectors of $\mathbf{F}$. That is, δ_i satisfy $\mathbf{F}'\boldsymbol{\delta}_i = 0$ and $\mathbf{b}'_k\boldsymbol{\delta}_i = 0$. Therefore,

$$\boldsymbol{\delta}'\overline{\mathbf{B}}\mathbf{w}_k = 0$$

where $\boldsymbol{\delta} = [\boldsymbol{\delta}_1 \ldots \boldsymbol{\delta}_{\ell_4}]$ and ℓ_4 is the dimension of the null space of $\mathbf{F}'$. Clearly, $\mathbf{w}_k$ reside in the null space of $\boldsymbol{\delta}'\overline{\mathbf{B}}$ with basis $[\overline{\mathbf{w}}_1, \overline{\mathbf{w}}_2, \ldots, \overline{\mathbf{w}}_{\ell_5}]$. The dimension of the matrix $\boldsymbol{\delta}'\overline{\mathbf{B}}$ is $\ell_4 \times 2m$ and $\overline{\mathbf{w}}_i$ are $2m$-component column vectors spanning the null space of $\boldsymbol{\delta}'\overline{\mathbf{B}}$. Suppose,

$$\mathbf{w}_k = d_1\overline{\mathbf{w}}_1 + d_2\overline{\mathbf{w}}_2 + \cdots + d_{\ell_5}\overline{\mathbf{w}}_{\ell_5}$$
$$= \overline{\mathbf{w}}d$$

where d_i are any non-zero constants. The eigenvalue–eigenvector constraint suggests

$$\overline{\mathbf{B}}\mathbf{w}_k = \mathbf{F}\mathbf{v}_k \Rightarrow \left[\overline{\mathbf{B}}\overline{\mathbf{w}} - F\tilde{\boldsymbol{\eta}}\right]\begin{bmatrix}\mathbf{d}\\ \mathbf{c}\end{bmatrix} = 0 \tag{7}$$

That is, the constants $\mathbf{d}$ and $\mathbf{c}$ are linearly dependent and they reside in the null space of $\left[\overline{\mathbf{B}}\overline{\mathbf{w}} - F\tilde{\boldsymbol{\eta}}\right]$. Therefore, the non-zero vectors $\{\mathbf{v}_k, \mathbf{w}_k\}$ and $\{\hat{\mathbf{v}}_i, \hat{\mathbf{w}}_i\}$ are exactly determined from Eqs. (6) and (7). Based on these vectors, the controller $\mathbf{G}$ is determined from Eq. (3). The procedure is given in the following section.

4. State and output feedback controller design

The controller for PEA is determined from the following relations:

$$\mathrm{GC}\hat{\mathbf{v}}_{i,R} = \hat{\mathbf{w}}_{i,R}$$
$$\mathrm{GC}\hat{\mathbf{v}}_{i,I} = \hat{\mathbf{w}}_{i,I}, \quad i = 1,2,\ldots,\overline{p} \tag{8a}$$

$$\mathrm{GC}\hat{\mathbf{v}}_{i,R} = \hat{\mathbf{w}}_{i,R}$$
$$\mathrm{GC}\hat{\mathbf{v}}_{i,I} = \hat{\mathbf{w}}_{i,I}, \quad i = 1,2,\ldots,\overline{p}+1, \quad \overline{p}+1,\ldots,2n \tag{8b}$$

Further the above equations resemble

$$\mathbf{I}_{(m)}\mathrm{G}\mathrm{v} = \mathbf{w} \tag{9}$$

where $\mathbf{I}_{(m)}$ is the identity matrix of order m, $\mathbf{G}$ is an unknown matrix of order $m \times p$and $\mathbf{v}$ and $\mathbf{w}$ are known non-zero vectors with components p and m, respectively. Let,

$$\mathbf{G} = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,p} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,p} \\ \cdots & \cdots & \cdots & \cdots \\ g_{m,1} & g_{m,2} & \cdots & g_{m,p} \end{bmatrix} \quad \text{and}$$

$$\mathbf{g}'_p = [g_{1,1}, g_{1,2}, \ldots, g_{1,p}, g_{2,1}, g_{2,2}, \ldots, g_{2,p}, \ldots, g_{m,1}, g_{m,2}, \ldots, g_{m,p}]$$

where prime denotes transpose of the column vector $\mathbf{g}_p$. Under this arrangement, Eq. (9) is modified as below:

$$[\mathbf{I}_{(m)} \otimes \mathbf{v}']\mathbf{g}_p = \mathbf{w} \tag{10}$$

where $\otimes$ refers the Kronecker product of the matrix $\mathbf{I}_{(m)}$ and the row vector $\mathbf{v}'$.

State feedback case:

In this case, matrix $\mathbf{C}$ is identity matrix of order $2n$. $\mathbf{g}_p$=$\mathbf{g}$. Eqs. (8) turns out to be

$$[\mathbf{I}_{(m)} \otimes \hat{\mathbf{v}}'_{i,R}]\mathbf{g} = \hat{\mathbf{w}}_{i,R}$$
$$[\mathbf{I}_{(m)} \otimes \hat{\mathbf{v}}'_{i,I}]\mathbf{g} = \hat{\mathbf{w}}_{i,I}, \quad i = 1,2,\ldots,\overline{p}$$

$$[\mathbf{I}_{(m)} \otimes \hat{\mathbf{v}}'_{i,R}]\mathbf{g} = \hat{\mathbf{w}}_{i,R}$$
$$[\mathbf{I}_{(m)} \otimes \hat{\mathbf{v}}'_{i,I}]\mathbf{g} = \hat{\mathbf{w}}_{i,I}, \quad i = 1,2,\ldots,\overline{p}+1, \quad \overline{p}+2,\ldots,2n$$

These equations are rewritten in the matrix form as

$$\boldsymbol{\Phi}g = \phi$$

where

$$\mathbf{\Phi}=\begin{bmatrix}\mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{1,R}\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{1,I}\\ \cdots\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{\overline{p},R}\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{\overline{p},I}\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{\overline{p}+1,R}\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{\overline{p}+1,I}\\ \cdots\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{n,R}\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{n,I}\end{bmatrix}\text{ and }\phi=\begin{bmatrix}\hat{\mathbf{w}}_{1,R}\\ \hat{\mathbf{w}}_{1,I}\\ \cdots\\ \hat{\mathbf{w}}_{\overline{p},R}\\ \hat{\mathbf{w}}_{\overline{p},I}\\ \mathbf{w}_{\overline{p}+1,R}\\ \mathbf{w}_{\overline{p}+1,I}\\ \cdots\\ \mathbf{w}_{n,R}\\ \mathbf{w}_{n,I}\end{bmatrix}$$

$\mathbf{\Phi}$ is a square matrix of order $2mn\times 2mn$. φ is a $2mn$-component vector. Thus the state feedback controller $\mathbf{G}$ is uniquely determined if $\mathbf{\Phi}$ is non-singular.

Output feedback case:

In this case, $\mathbf{C}$ is a matrix of order $p\times n$. Eq. (8) is given by

$$\mathbf{\Psi}\mathbf{g}_p=\varphi$$

$$\mathbf{\Psi}=\begin{bmatrix}\mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{1,R}\mathbf{C}'\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{1,I}\mathbf{C}'\\ \cdots\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{\overline{p},R}\mathbf{C}'\\ \mathbf{I}_{(m)}\otimes\hat{\mathbf{v}}'_{\overline{p},I}\mathbf{C}'\\ \mathbf{I}_{(m)}\otimes\mathbf{v}'_{\overline{p}+1,R}\mathbf{C}'\\ \mathbf{I}_{(m)}\otimes\mathbf{v}'_{\overline{p}+1,I}\mathbf{C}'\\ \cdots\\ \mathbf{I}_{(m)}\otimes\mathbf{v}'_{n,R}\mathbf{C}'\\ \mathbf{I}_{(m)}\otimes\mathbf{v}'_{n,I}\mathbf{C}'\end{bmatrix}$$

$\mathbf{\Psi}$ is a square matrix of order $2mn\times mp$. Further $p<2n$ suggests that the number of equations are more than the number of unknowns to be solved. Solution will exist if the rows of $[\mathbf{\Psi}\quad\varphi]$ are linearly dependent. Otherwise exact solution for $\mathbf{g}_p$ is not guaranteed. Note that the approximate solution in the least squared error minimization sense may not guarantee poles in the open left half plane for stability or exact pole placement in the left half plane.

5. Examples

Consider a spring–mass system in Fig. 1 with mass, stiffness and damping matrices given by

$$\mathbf{M}=\begin{bmatrix}m_1&0&0\\0&m_2&0\\0&0&m_3\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}e_1+e_2&-e_2&0\\-e_2&e_2+e_3&-e_3\\0&-e_3&e_3+e_4\end{bmatrix}\text{ and }\mathbf{K}=\begin{bmatrix}k_1+k_2&-k_2&0\\-k_2&k_2+k_3&-k_3\\0&-k_3&k_3+k_4\end{bmatrix}$$

The control influence matrix $\mathbf{D}$ is given by

$$\mathbf{D}=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$$

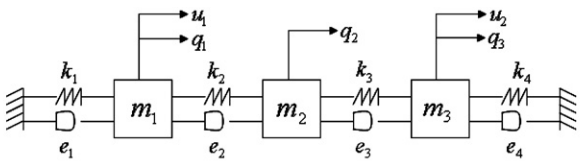

Fig. 1. Model for structural damage mitigation using PEA.

If spring k_2 is damaged, then $k_2 \to k_2 + \Delta k_2$. Accordingly, $\mathbf{K} \to \mathbf{K} + \alpha_1 \mathbf{K}_1$, where $\alpha_1 = \Delta k_2$ and

$$\mathbf{K}_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The controllable pair ($\hat{\mathbf{A}}$,$\mathbf{B}$) for the damaged structure becomes

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -21 & 11 & 0 & -0.6 & 0.4 & 0 \\ 5.5 & -15.5 & 10 & 0.2 & -0.4 & 0.2 \\ 0 & 6.6667 & -13.3333 & 0 & 0.1333 & -0.2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/3 \end{bmatrix}$$

The following numerical values are assumed in the above matrices: $m_1 = 1$, $m_2 = 2$ and $m_3 = 3$. Further, $k_1 = k_2 = k$, $k_3 = k_4 = 2k$ where $k = 1$. Similarly, $e_1 = e_4 = e$, $e_2 = e_3 = 2e$ where $e = 0.2$. When $\alpha_1 = 1$, the eigenvalues of the damaged structure are given by,

$$\lambda_{1,2} = -0.03291228226861 + 2.09664357437356j$$
$$\lambda_{3,4} = -0.39515341479565 \pm 5.30006344031241j$$
$$\lambda_{5,6} = -0.17193430293574 + 4.14087801817963j$$

Since $\lambda_{1,2}$ exhibits least damping, we are interested in modifying the eigenvalue $\lambda_{1,2}$ and enhance damping and stiffness. Therefore, assume $\hat{\lambda}_{1,2} = -0.1111 \pm 3j$. Thus, the desired eigenvalues for PEA are $\hat{\lambda}_{1,2}$, $\lambda_{3,4}$ and $\lambda_{5,6}$. Following the procedure discussed in Section 3, the vectors required to compute the gain matrix $\mathbf{G}$ are

$$\hat{\mathbf{w}}_{1,R} = \begin{bmatrix} -1.19652791651476 \\ -7.60259117976494 \end{bmatrix}, \quad \hat{\mathbf{w}}_{1,I} = \begin{bmatrix} -40.717336288475 \\ 388.860309567509 \end{bmatrix}$$

$$\hat{\mathbf{v}}_{1,R} = \begin{bmatrix} 1.14113819790217 \\ 1.07651211901762 \\ 0.37036666666667 \\ -65.56551835765345 \\ -60.32310248704926 \\ -3.04114773666667 \end{bmatrix}, \quad \hat{\mathbf{v}}_{1,I} = \begin{bmatrix} 21.81291263462217 \\ 20.06783399687546 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_{3,R} = \begin{bmatrix} 2.10441368560734 \\ 3.41138583103054 \end{bmatrix}, \quad \mathbf{w}_{3,I} = \begin{bmatrix} 19.64290399946321 \\ -67.91590325654130 \end{bmatrix}$$

$$\mathbf{v}_{3,R} = \begin{bmatrix} 1.03750625322858 \\ -0.54920306004516 \\ 0.26323334248871 \\ -60.75156303393706 \\ 30.47116247565759 \\ -5.40408099448490 \end{bmatrix}, \quad \mathbf{v}_{3,I} = \begin{bmatrix} 11.38506917410504 \\ -5.70826054287403 \\ 1 \\ 1 \\ -0.65517241379310 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_{5,R} = \begin{bmatrix} 0.28978427549367 \\ -0.17312563294820 \\ 0.28301589609513 \\ -4.86570694921010 \\ 41.37951873026788 \\ -4.18953815899448 \end{bmatrix}, \quad \mathbf{v}_{5,I} = \begin{bmatrix} 1.16301013230968 \\ -9.98574512789546 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_{5,R} = \begin{bmatrix} 14.78567599941380 \\ 14.54403997030568 \end{bmatrix}, \quad \mathbf{w}_{5,I} = \begin{bmatrix} -114.146175933530 \\ -187.354000242387 \end{bmatrix}$$

Following the procedure discussed in Section 4, these vectors are used to compute the controller $\mathbf{G}$ as below:

$$\mathbf{G}' = \begin{bmatrix} 16.674829756373 & 5.197227254444 \\ -9.014060120494 & 15.601871303801 \\ -195.646829611661 & -34.288586266912 \\ 1.415773491085 & 0.207873113289 \\ -1.145191189470 & 0.256789479249 \\ -28.175031515465 & -3.778194166868 \end{bmatrix}$$

We observe that both ($\hat{\mathbf{A}}$−BG) and $\hat{\mathbf{A}}$ share common eigenvalues as unaltered under a state feedback control law $u(t)=-\mathbf{G}x(t)$. The altered eigenvalue is at $-0.1111\pm 3j$. It illustrates damping and stiffness modifications required in structural damage mitigation problems.

In the next example, partial eigenvalue assignment for a cantilever beam model is illustrated [17]. It is represented by ten states and two inputs. The matrices are given by

$$\hat{\mathbf{A}}=\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2.334e7 & 3.465e7 & -1.938e7 & 2.734e6 & 1.097e6 & -24.42 & 14.05 & 1.241 & -8.169 & -1.856 \\ 5.653e7 & -1.204e8 & 1.180e8 & -4.236e7 & -4.915e6 & 24.42 & -47.55 & 19.19 & -1.528 & 2.264 \\ -3.016e7 & 1.09e8 & -2.429e8 & 1.656e8 & -1.290e7 & 5.013 & 21.73 & -66.56 & 32.01 & 4.129 \\ -4.608e6 & -1.090e7 & 1.425e8 & -1.799e8 & 5.264e7 & 3.169 & 4.390 & 26.19 & -51.46 & 13.49 \\ 6.401e6 & -6.459e6 & -9.703e7 & 1.772e8 & -7.933e7 & -5.384 & 7.757 & -2.545 & 37.50 & -43.32 \end{bmatrix}\right)$$

$$\mathbf{B}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1503 & -3937 \\ 7544 & 2.087e4 \\ -1.475e4 & -4.326e4 \\ 1.006e4 & 3.130e4 \\ -8.576e4 & -2.738e4 \end{bmatrix}$$

Eigenvalues of the matrix $\hat{\mathbf{A}}$ are $\lambda_{ij}e+04$, where λ_{ij} are given by

$$\lambda_{1,2}=-0.00483536697679\pm 2.07034435005781i$$
$$\lambda_{3,4}=-0.00309197453926\pm 1.28828185293888i$$
$$\lambda_{5,6}=-0.00252556153054\pm 0.66745711476861i$$
$$\lambda_{7,8}=-0.00072174022177\pm 0.23074774058783i$$
$$\lambda_{9,10}=-0.00049085673164\pm 0.11798597206295i$$

Suppose the eigenvalues $\lambda_{9,10}e+04$ need to be modifies to

$$\hat{\lambda}_{9,10}=-5.89028077963392\pm 1.297845692692489e+003$$

Then the controller is $\mathbf{G}\,e+03$, where $\mathbf{G}$ is given by

$$\mathbf{G}'=\begin{bmatrix} -0.00137223984125 & -1.57922757003882 \\ -0.00080957309888 & -1.00824516274194 \\ -0.00047006997748 & -0.69745461280534 \\ -0.00072650796522 & -0.84102125257781 \\ -0.00025813148110 & -0.41108566068630 \\ -0.00000030016279 & 0.00002527023762 \\ -0.00000018822599 & 0.00001663196419 \\ -0.00000012610088 & 0.00001184395892 \\ -0.00000015317961 & 0.00001479883008 \\ -0.00000007203303 & 0.00000738001074 \end{bmatrix}$$

The modified eigenvalues of the matrix ($\hat{\mathbf{A}}$−BG) are $\lambda_{i,j}e+04$and $\hat{\lambda}_{i,j}e+04$ where

$$\lambda_{1,2}\approx -0.00483536697679\pm 2.07034435005773i$$
$$\lambda_{3,4}\approx -0.00308964884495\pm 1.28827443360098i$$
$$\lambda_{5,6}\approx -0.00252556153054\pm 0.66745711476842i$$
$$\lambda_{7,8}\approx -0.00072174022176\pm 0.23074774058759i$$
$$\hat{\lambda}_{9,10}=-0.00058902807797\pm 0.12978456926916i.$$

The altered and unaltered mode-wise damping and frequency of oscillation are given in Table 1.

Table 1
Basic principle in structural damage mitigation.

	Damping factor ς, frequency ω rad/s	
	Without feedback	With feedback
Mode 1	ς=2.34e−003, ω=2.07e+004	ς=2.34e−003, ω=2.07e+004
Mode 2	ς=2.40e−003, ω=1.29e+004	ς=2.40e−003, ω=1.29e+004
Mode 3	ς=3.78e−003, ω=6.67e+003	ς=3.78e−003, ω=6.67e+003
Mode 4	ς=3.13e−003, ω=2.31e+003	ς=3.13e−003, ω=2.31e+003
Mode 5	ς=4.16e−003, ω=1.18e+003	ς=4.54e−003, ω=1.30e+003 (modified)

6. Concluding remarks

In this paper, linear algebraic methods for partial eigenvalue assignment by a state or an output feedback controller using first order systems are presented. As a potential application of these controllers, structural damage mitigation problem is illustrated. Partial eigenvalue assignment in structural damage mitigation augments necessary damping and/or stiffness properties to a set of selective eigenmodes where such properties are deteriorated due to damage. Two examples are illustrated to demonstrate the partial eigenvalue assignment methodology.

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